HW 4

P1.)

$\langle 1,1\rangle = \int_{-1}^{1} 1\,dx = 2$
$\langle x,1\rangle = \int_{-1}^{1} x\,dx = 0$
$\langle x^2,1\rangle = \int_{-1}^{1} x^2 dx = 2/3$
$\langle x^2,x\rangle = \int_{-1}^{1} x^3 dx = 0$

A.) $P_0(x) = 1$

$$P_1(x) = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}\cdot 1 = x - 0 = x$$

$$P_2(x) = x^2 - \frac{\langle x^2,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle}1$$

$$= x^2 - 1/3$$

B.) $P_m\left[\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x)\right] = 0$

$- P_n\left[\left[(1-x^2)P_m'(x)\right]' + m(m+1)P_m(x)\right] = 0$

$\Rightarrow P_m\left[(1-x^2)'P_n'(x) + (1-x^2)P_n''(x)\right] + \left[n(n+1) - m(m+1)\right]P_nP_m$
$- (1-x^2)'P_nP_m' - (1-x^2)P_nP_m'' = 0$

$\Rightarrow -\left[(1-x^2)(P_m'P_n - P_n'P_m)\right]' - (m-n)(m+n+1)P_nP_m = 0$

C.) $$\int_{-1}^{1} P_nP_m\,dx = \int_{-1}^{1}\left[(1-x^2)(P_m'P_n - P_n'P_m)\right]'\left(\frac{-1}{(m-n)(m+n+1)}\right)dx$$

$$= \left[(1-x^2)(P_m'P_n - P_n'P_m)\left(\frac{-1}{(m-n)(m+n+1)}\right)\right]_{-1}^{1} = 0$$

P2.)

A.) At $x = \pi/2$, the Fourier Series gives $1/2$
but $h(\pi/2) = 0$

B.) For $x \neq \pi/2 + 2\pi\mathbb{Z}$, $\dfrac{h(x) - h(x-\varepsilon)}{\varepsilon} = 0$ for small enough ε (for $\varepsilon < \min\limits_{k\in\mathbb{Z}} |x - \pi/2 - \pi k|$)

P4). A.) $\left| e^{-ikx} - q_{m,k}(x) \right| \le \sum_{m+1}^{\infty} \frac{k^l |x|^l}{l!} \le \sum_{m+1}^{\infty} \frac{k^l \pi^l}{l!}$

For large m, this sum ~~[illegible]~~ may be taken to be arbitrarily small.

B.) $\langle q_{m,k}, f \rangle = \sum_{l=0}^{m} \frac{(-ik)^l}{l!} \int_{-\pi}^{\pi} x^l f(x)\,dx = \sum 0 = 0$

C.) $\langle f, e^{-ikx} \rangle = \lim \langle f, q_{m,k} \rangle = 0$

P3.) A.)

$|x|$ $\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \pi/2$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \pm x e^{-ikx} dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x e^{-ikx} dx + \int_{0}^{\pi} x e^{-ikx} dx\right]$$

$$\int_{-\pi}^{0} -x e^{-ikx} dx = \frac{-x e^{-ikx}}{-ik}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} \frac{-1}{-ik} e^{-ikx} dx$$

$$= \frac{\pi e^{ik\pi}}{ik} - \left[\frac{e^{-ikx}}{+k^2}\Big|_{-\pi}^{0}\right]$$

$$= \frac{\pi e^{ik\pi}}{ik} - \left[-1/k^2 + \frac{e^{ik\pi}}{-k^2}\right]$$

$$\int_{0}^{\pi} x e^{-ikx} dx = \frac{x e^{-ikx}}{-ik}\Big|_{0}^{\pi} - \int_{0}^{\pi} \frac{1}{-ik} e^{-ikx} dx$$

$$= \frac{\pi e^{-ik\pi}}{-ik} - \left[1/k^2\, e^{-ikx}\Big|_{0}^{\pi}\right] = \frac{\pi e^{-ik\pi}}{-ik} + \frac{e^{-ik\pi}}{k^2} - 1/k^2$$

So $\int_{-\pi}^{\pi} |x| e^{-ikx} dx = -2/k^2 + 2\cos(k\pi)/k^2$

$$= \begin{cases} \frac{-2}{k^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

and the series is $\pi/2 - \frac{1}{2\pi}\sum_{\substack{k \text{ odd} \\ k\in\mathbb{Z}}} 4/k^2\, e^{ikx}$

$$= \pi/2 - 4/\pi \sum_{\substack{k \text{ odd} \\ k\in\mathbb{N}}} \frac{1}{k^2}\cos(kx)$$

B.) Yes, by thm. 8.3

$$\pi/2 - 4/\pi \sum_{\substack{k \text{ odd} \\ k\in\mathbb{N}}} 1/k^2 = 0$$

$$\Leftrightarrow \quad \pi^2/8 = \sum_{\substack{k \text{ odd} \\ k\in\mathbb{N}}} 1/k^2$$

C.) This is Parseval's Identity. : $\frac{1}{2\pi}\|x\|_2^2 = \pi^2/3 = \pi^2/4 + \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} \frac{4}{\pi^2 k^4}$

$\Leftrightarrow$ ~~[illegible]~~ $\pi^4/96 = \sum_{\substack{k \text{ odd} \\ k\in\mathbb{N}}} \frac{1}{k^4}$

~~$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \pi^2/3$ which $= \pi^2/2$~~